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## LETTER TO THE EDITOR

### WKB equivalent potentials for the $q$ -deformed harmonic oscillator

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**Abstract.** WKB equivalent potentials (WKB-EP) giving the same spectrum as the  $q$ -deformed harmonic oscillator with the symmetry  $SU_q(2)$  are determined. While in the case of  $q$  being real the WKB-EP goes to infinity as one moves away from the origin, in the case of  $q$  being a phase the WKB-EP goes to a finite limiting value, thus resembling, for example, the modified Pöschl-Teller potential.

Quantum algebras [1-4] have been attracting much attention recently. In particular, the realization of the quantum algebra  $SU_q(2)$  in terms of the  $q$ -analogue of the quantum harmonic oscillator [5-7] has initiated much work on this topic [8-12]. In addition, the quantum rotor, also having the symmetry  $SU_q(2)$ , has been found useful in the description of rotational spectra of deformed nuclei [13, 14], superdeformed nuclei [15] and diatomic molecules [16], while vibrational spectra of diatomic molecules can be described in terms of the  $q$ -deformed anharmonic oscillator [17].

In the case of the  $q$ -analogue of the quantum harmonic oscillator [5-12], the creation and annihilation operators  $a^+$  and  $a$  satisfy the commutation relation

$$aa^+ - q^{-1}a^+a = q^N \quad (1)$$

where  $N$  is the number operator, satisfying

$$[N, a^+] = a^+ \quad [N, a] = -a. \quad (2)$$

The relevant Fock space is defined as

$$a|0\rangle = 0 \quad |n\rangle = \frac{(a^+)^n}{([n]!)^{1/2}}|0\rangle \quad (3)$$

where the  $q$ -factorial is defined as

$$[n]! = [n][n-1] \dots [1] \quad (4)$$

and the  $q$ -numbers are defined by

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (5)$$

$q$  can be either real ( $q = e^\tau$ ), in which case the  $q$ -numbers take the form

$$[x] = \frac{\sinh(\tau x)}{\sinh(\tau)} \quad (6)$$

or a phase ( $q = e^{i\tau}$ ), in which case the  $q$ -numbers are

$$[x] = \frac{\sin(\tau x)}{\sin(\tau)}. \tag{7}$$

It is clear that in both cases  $[x] \rightarrow x$  in the limit  $q \rightarrow 1$ .

The Hamiltonian of the  $q$ -harmonic oscillator is

$$H = \frac{\hbar\omega}{2} (aa^+ + a^+a). \tag{8}$$

Its eigenvalues in the Fock space defined above are then

$$E(n) = \frac{\hbar\omega}{2} ([n] + [n + 1]). \tag{9}$$

The same spectrum is obtained by solving the  $q$ -generalization of the Schrödinger equation for the classical potential of the harmonic oscillator [18]. A different approach is to find the potential which would yield through the usual Schrödinger equation the same wkb energy spectrum (equation (9)) as the  $q$ -harmonic oscillator. This can be achieved through use of standard inverse spectrum techniques [19, 20]. The result should make the physical meaning of  $q$ -deformation clearer and will give a physical analogue compatible with the classical quantum theory.

Consider a symmetric potential  $V(x)$ , with an absolute minimum at  $x = 0$  and let  $V(x)$  be a non-decreasing function for positive values of  $x$ ,

$$V(x) = V(-x) \quad xV'(x) \geq 0 \quad \min V(x) = E_{\min}. \tag{10}$$

It should be kept in mind that  $E_{\min}$  denotes the energy  $E(x)$  at the position  $x$  at which the potential  $V(x)$  is minimum. (We keep this notation since it is customary in the relevant inverse spectrum literature [19, 20].)

The first-order wkb approximation of the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x)$$

defines the wkb energy spectrum

$$(n + \frac{1}{2})\pi = 2\left(\frac{2m}{\hbar^2}\right)^{1/2} \int_0^x (E - V(x'))^{1/2} dx' \tag{11}$$

where  $x$  is the turning point satisfying the relation

$$E = V(x) \quad \text{and} \quad E_{\min} = V(0). \tag{12}$$

The equation  $n = n(E)$  defined by (11) is an increasing function of the parameter  $E$  satisfying the condition

$$n(E_{\min}) + \frac{1}{2} = 0$$

if conditions (10) are valid.

Following Wheeler [20] or Chadan and Sabatier [19] we can calculate, without ambiguity, the potential  $V(x)$  if the spectral function

$$n = n(E) \tag{13}$$

is defined. The conditions imposed to the potential imply that the energy spectrum functions  $E(n)$  and  $n(E)$  in equation (13) are uniquely defined in the framework of the present method. (This does not mean, however, that the solution to the inverse spectrum problem to be found here is the only possible one. Additional solutions might exist, which could be found through different methods.) The method is summarized as follows. The 'inclusion'

$$I(E) = 2 \int_0^x (E - V(x')) dx' \quad (14)$$

is given using the inverse Abel transform

$$I(E) = \left( \frac{\hbar^2}{2m} \right)^{1/2} \int_{E_{\min}}^E (E - E')^{-1/2} (n(E') + \frac{1}{2}) dE'. \quad (15)$$

From the inclusion (15) we can define the 'excursion' for the symmetric potential  $V(x)$ :

$$X(E) = \frac{\partial I}{\partial E} = 2x. \quad (16)$$

The function  $X(E)$  is a known function of the energy if the spectrum (13) is known. From equations (12) and (16) we can calculate the function  $V(x)$  which corresponds to the WKB equivalent potential (WKB-EP) to the spectrum (13), i.e. to a potential whose WKB spectrum coincides with the known spectrum (13).

From (6), (7) and (9) we find that the  $q$ -oscillator has a spectrum given by

$$E_n = \frac{\hbar\omega}{2} \frac{\sin(\tau(n + \frac{1}{2}))}{\sin(\tau/2)} \quad (17)$$

when  $q$  is a phase ( $q = e^{i\tau}$ ), while in the case that  $q$  is real ( $q = e^\tau$ ) the spectrum is

$$E_n = \frac{\hbar\omega}{2} \frac{\sinh(\tau(n + \frac{1}{2}))}{\sinh(\tau/2)}. \quad (18)$$

In what follows, the two cases will be examined separately.

We first consider the more complicated case in which  $q$  is a phase. In this case the spectrum is defined by (17). If  $q$  is a root of the unity then  $\tau$  is a rational multiplier of  $2\pi$ , i.e.

$$\tau = (k/l)2\pi, \quad k \text{ and } l \text{ are natural numbers with } 0 \leq k < l$$

and the great common divisor of  $k$  and  $l$  is 1.

In this case the energy eigenvalues given by the spectrum (17) have discrete values, such that

$$|E_n| \leq E_{\max} = \frac{\hbar\omega}{2} \frac{1}{|\sin(\tau/2)|}. \quad (19)$$

We must point out that, if  $\tau$  is a non-rational multiplier of  $2\pi$ , then there is a set of energy eigenvalues densely distributed inside the energy interval defined by (19), i.e. the eigenvalues constitute a band and there is no potential satisfying the general assumptions (10). The assumption that  $\tau$  should be a rational multiplier of  $2\pi$  is quite common in many papers concerning the quantum group theory [21].

The energy formula (17) could be used for fitting experimental data of vibrational states, in a way similar to that pointed out in [17]. In such cases, however, the number of levels to be fitted is relatively small, so that the quantum number  $n$  acquires only the first few lower values (0, 1, 2, ...) and the energy satisfies the inequality

$$0 \leq E_n \leq E_{\max}.$$

(See [13-16] for additional similar examples.) In this case we have that

$$E_{\min} = 0$$

and

$$n(E) + \frac{1}{2} = \frac{1}{\tau} \sin^{-1} \left( \frac{2E}{\hbar\omega} \sin(\tau/2) \right).$$

It should be recalled here that  $E_{\min}$  is the value of the energy at the position at which the potential  $V(x)$  is minimum, and not the minimum value of the energy of the oscillator.

The inclusion (14) is calculated by

$$I(E) = \frac{1}{\tau} \left( \frac{\hbar^2}{2m} \right)^{1/2} \int_0^E (E - E')^{-1/2} \sin^{-1} \left( \frac{2E'}{\hbar\omega} \sin(\tau/2) \right) dE'. \quad (20)$$

This integral after an integration by parts is transformed into the form

$$I(E) = \frac{2}{\tau} \left( \frac{\hbar^2}{2m} \right)^{1/2} \left( \frac{\hbar\omega}{2 \sin(\tau/2)} \right)^{1/2} J(s) \quad (21)$$

where

$$s = \frac{2E \sin(\tau/2)}{\hbar\omega} \quad (22)$$

and

$$J(s) = \int_0^s \frac{(s-t)^{1/2}}{(1-t^2)^{1/2}} dt. \quad (23)$$

The excursion function given by (16) is

$$X(E) = \frac{\partial I}{\partial E} = \frac{2}{\tau} \left( \frac{\hbar^2}{2m} \right)^{1/2} \left( \frac{2 \sin(\tau/2)}{\hbar\omega} \right)^{1/2} \frac{\partial J}{\partial s} \quad (24)$$

which gives

$$x = \frac{1}{\tau} \left( \frac{\hbar^2}{2m} \right)^{1/2} \left( \frac{2 \sin(\tau/2)}{\hbar\omega} \right)^{1/2} \int_0^s \frac{dt}{(s-t)^{1/2} (1-t^2)^{1/2}}. \quad (25)$$

The integral can be reduced in an elliptic integral of the first kind

$$\int_0^s \frac{dt}{(s-t)^{1/2} (1-t^2)^{1/2}} = \sqrt{2} F(\delta, M) = \sqrt{2} \int_0^\delta \frac{da}{(1 - M^2 \sin^2 a)^{1/2}} \quad (26)$$

where

$$\delta = \sin^{-1} \left( \sqrt{\frac{2s}{s+1}} \right) \quad \text{and} \quad M = \sqrt{\frac{s+1}{2}}. \quad (27)$$

From  $E = V(x)$  we can invert (25) numerically and calculate the function  $V = V(x)$ . The form of the potential  $V(x)$  is shown in figure 1 for various values of the parameter  $\tau$ . For  $\tau = 0$  the usual harmonic oscillator potential

$$V(x) = \frac{m\omega^2}{2} x^2$$

is obtained. For  $\tau \neq 0$  it is clear that the potential reaches a finite limiting value as  $x \rightarrow \infty$  given by equation (19):

$$\lim V(x) = \frac{\hbar\omega}{2 \sin(\tau/2)}. \tag{28}$$

We consider next the case in which  $q$  is real. In this case the spectrum is defined by

$$n(E) + \frac{1}{2} = \frac{1}{\tau} \sinh^{-1} \left( \frac{2E}{\hbar\omega} \sinh(\tau/2) \right). \tag{29}$$

The inclusion (14) is calculated by

$$I(E) = \frac{1}{\tau} \left( \frac{\hbar^2}{2m} \right)^{1/2} \int_0^E (E - E')^{-1/2} \sinh^{-1} \left( \frac{2E'}{\hbar\omega} \sinh(\tau/2) \right) dE'. \tag{30}$$

This integral, after an integration by parts, is transformed into

$$I(E) = \frac{2}{\tau} \left( \frac{\hbar^2}{2m} \right)^{1/2} \left( \frac{\hbar\omega}{2 \sin(\tau/2)} \right)^{1/2} G(s) \tag{31}$$

where

$$s = \frac{2E \sinh(\tau/2)}{\hbar\omega} \tag{32}$$

and

$$G(s) = \int_0^s \frac{(s-t)^{1/2}}{(1+t^2)^{1/2}} dt. \tag{33}$$

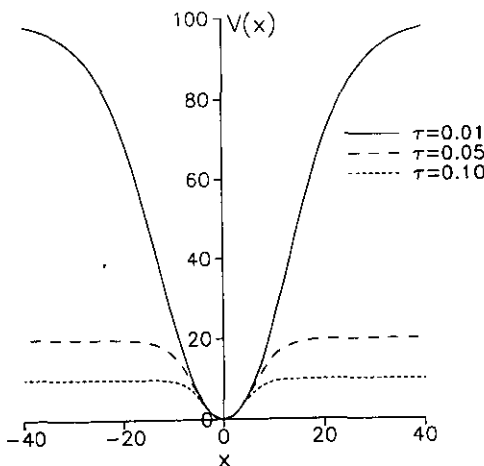


Figure 1. WKB potentials giving the same spectrum as the  $q$ -deformed harmonic oscillator with the symmetry  $SU_q(2)$  for  $q$  being a phase ( $q = \exp(i\tau)$ ) are given for various values of  $\tau$ . (The parameters are  $\hbar\omega = 1$  and  $\hbar^2/2m = 1$ .)

The excursion function given by (16) is

$$x = \frac{1}{\tau} \left( \frac{\hbar^2}{m} \right)^{1/2} \left( \frac{2 \sinh(\tau/2)}{\hbar \omega} \right)^{1/2} \frac{\partial G}{\partial s}. \quad (34)$$

The function  $\partial G/\partial s$  is calculated using the generalized hypergeometric function  ${}_3F_2$

$$x = \left( \frac{2}{m\omega^2} \right)^{1/2} \frac{2 \sinh(\tau/2)}{\tau} E^{1/2} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{3}{4}, \frac{5}{4}; -s^2\right). \quad (35)$$

From  $E = V(x)$  we can invert this equation numerically and calculate the function  $V = V(x)$ . The form of the potential  $V(x)$  is shown in figure 2 for various values of the parameter  $\tau$ . For  $\tau=0$  the usual harmonic oscillator potential is obtained. For  $\tau \neq 0$  it is clear that the potential goes to infinity for  $x \rightarrow \infty$ , as in the case of the usual harmonic oscillator potential.

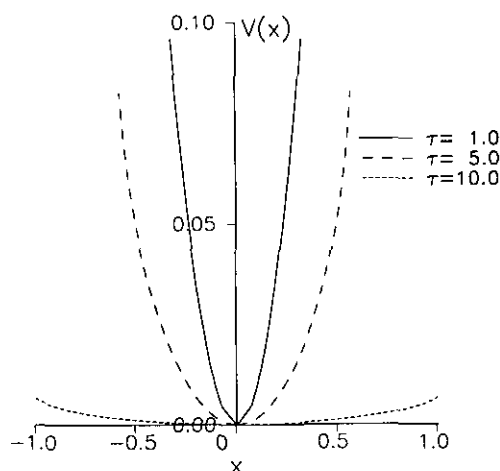


Figure 2. wKB potentials giving the same spectrum as the  $q$ -deformed harmonic oscillator with the symmetry  $SU_q(2)$  for  $q$  real ( $q = \exp(\tau)$ ) are given for various values of  $\tau$ . (The parameters are the same as in figure 1.)

In summary, we have determined the wKB-EP which gives the same spectrum as the  $q$ -harmonic oscillator, according to the method of Chadan and Wheeler [19, 20]. In the case of real  $q$ , the potential goes to infinity as  $x \rightarrow \infty$ , while in the case of  $q$  being a phase the potential goes to a finite value as  $x \rightarrow \infty$ . The latter case corresponds to a harmonic oscillator with squeezed spectrum. The potential resulting in the case of  $w$  being a phase is more similar to the Pöschl-Teller potential [22] (which has been recently used in hypernuclear physics [23]) and to the Woods-Saxon potential [24] (which is widely used in nuclear physics) than to the classical harmonic oscillator potential. It is therefore of interest to determine the wKB-EP giving the same spectrum as the  $q$ -rotor [13-16], which has been found suitable for the description of rotational spectra of deformed nuclei [13, 14], superdeformed nuclei [15] and diatomic molecules [16], as well as the wKB-EP giving the same spectrum as the  $q$ -anharmonic oscillator [17], which has been found appropriate for the description of vibrational spectra of diatomic molecules. It is worth mentioning that in all cases [13-17] it is the choice of  $q$  being a phase which gives the physically interesting result. In the case of the vibrational

spectra of diatomic molecules [17] it is of interest to examine the relation between the  $\text{wKB-EP}$  giving the same spectrum as the  $q$ -anharmonic oscillator [17] and the widely used Morse potential [25]. Work in these directions is in progress.

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## References

- [1] Kulish P P and Reshetikhin N Yu 1981 *Zapiski Semenovskogo LOMI* **101** 101
- [2] Sklyanin E K 1982 *Funct. Anal. Appl.* **16** 262
- [3] Drinfeld V G 1986 *Quantum Groups, Proc. Int. Congr. Math.*, ed A M Gleason (Providence, RI: Am. Math. Soc.) p 798
- [4] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
- [5] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
- [6] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
- [7] Jannussis A 1990 *Preprint* University of Patras
- [8] Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415
- [9] Ng Y J 1990 *J. Phys. A: Math. Gen.* **23** 1023
- [10] Ui H and Aizawa N 1990 *Mod. Phys. Lett. A* **5** 237
- [11] Song X C 1990 *J. Phys. A: Math. Gen.* **23** L821
- [12] Yan H 1990 *J. Phys. A: Nucl. Part. Phys.* **23** L1155
- [13] Raychev P P, Roussev R P and Smirnov Yu F 1990 *J. Phys. G: Nucl. Part. Phys.* **16** L137
- [14] Bonatsos D, Argyres E N, Drenska S B, Raychev P P, Roussev R P and Smirnov Yu F 1990 *Phys. Lett.* **251B** 477
- [15] Bonatsos D, Drenska S B, Raychev P P, Roussev R P and Smirnov Yu F 1991 *J. Phys. G: Nucl. Part. Phys.* **17** in press
- [16] Bonatsos D, Raychev P P, Roussev R P and Smirnov Yu F 1990 *Chem. Phys. Lett.* **175** 300
- [17] Bonatsos D, Raychev P P and Faessler A 1991 *Chem. Phys. Lett.* **178** 221
- [18] Minahan J A 1990 *Mod. Phys. Lett. A* **5** 2625
- [19] Chadan K and Sabatier P C 1977 *Inverse Problems in Quantum Scattering Theory* (Heidelberg: Springer) chapter XVI
- [20] Wheeler J A 1976 *Studies in Mathematical Physics: Essays in Honor of Valentine Bargmann* Princeton University
- [21] Baulieu L and Floratos E G 1990 *Preprint* Ecole Normale Supérieure de Paris, LPTENS 90/20 Aug. 90  
Floratos E G and Tomaras T N 1990 *Phys. Lett.* **251B** 164  
Arnaudon D and Chakrabarti 1991 *Phys. Lett.* **255B** 242  
Polychronakos A P 1990 *Mod. Phys. Lett. A* **5** 2325
- [22] Frank A and Wolf K B 1985 *J. Math. Phys.* **26** 973
- [23] Grypeos M, Lalazissis G and Massen S 1986 *Nucl. Phys. A* **450** 283c  
Lalazissis G A, Grypeos M E and Massen S E 1988 *Phys. Rev. C* **37** 2098  
Lalazissis G A, Grypeos M E and Massen S E 1989 *J. Phys. G: Nucl. Part. Phys.* **15** 303  
Grypeos M E, Lalazissis G A, Massen S E and Pancos C P 1989 *Z. Phys. A* **332** 391
- [24] Preston M A and Bhaduri R K 1975 *Structure of the Nucleus* (Reading, MA: Addison-Wesley)
- [25] Morse P M 1929 *Phys. Rev.* **34** 57